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FORCED CONVECTION IN WEDGE FLOW WITH NON-ISOTHERMAL SURFACES

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Abstract—The steady heat transfer across laminar, incompressible, constant property boundary layers over wedges with a step discontinuity in surface temperature is investigated. The analysis begins with an appropriate transformation of the energy boundary layer equation with the consequence that the non-similar solution of the problem decomposes into an infinite sequence of simple, similar solutions. Of significance is the fact that these similar solutions are expressible as universal functions and, thus, can be tabulated once and for all. For fluids with Prandtl number of the order of unity or larger, only a very few of the functions are needed to achieve results of high accuracy. A tabulation of such functions is given. With these, the determination of the temperature field in the boundary layer, as well as the local heat transfer rate at the wedge surface, becomes a matter of simple arithmetic. Similar information for wedges with any arbitrarily prescribed surface temperature distribution can likewise be obtained.

NOMENCLATURE

a, $f''(0)$;
a_n, coefficients in series (4), beginning with $n = 2$ and $a_2 \equiv a$;
b, $(aPr/3!)^{1/3}$;
c, $(3/2)(2 - \beta)^{-1}$;
c_p, specific heat;
k, thermal conductivity;
M, $3\beta/2ab$;
Pr, Prandtl number;
q_w, heat flux at wall;
Re, Reynolds number $= u_1 x/\nu$; for flat plate $u_1 = u_\infty$;
St, Stanton number $= q_w/c_p \rho u_1 (T_w - T_\infty)$; for flat plate $u_1 = u_\infty$;
T, temperature;
u, velocity component in x-direction;
v, velocity component in y-direction;
X, transformed dimensionless coordinate defined in (7a);
x, coordinate along wedge surface;
y, coordinate normal to wedge surface;
β, wedge angle divided by π;
$\Gamma(n)$, gamma function $= \int_0^\infty \alpha^{n-1} e^{-\alpha} d\alpha$;
$\Gamma(n,x)$, incomplete gamma function $= \int_0^x \alpha^{n-1} e^{-\alpha} d\alpha$;
η, dimensionless coordinate defined in (2);
θ, dimensionless temperature defined in (6);
κ, thermal diffusivity;
ν, kinematic viscosity;
ξ, transformed dimensionless coordinate defined in (7b);
ρ, density.

Subscripts

1, refers to edge of velocity boundary layer;
∞, refers to free stream;
w, refers to wedge surface.

1. INTRODUCTION

In many technological applications, heat transfer by convection takes place over surfaces which have a significant temperature variation in the direction of the main flow. This non-uniformity of temperature is often the consequence of

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design requirements. Rubesin [1, 2] was probably among the first to recognize its importance in the prediction of convective heat transfer rates. In the steady flight of an aircraft or other objects through the atmosphere, the boundary layer over the forward portion of surfaces is generally laminar. It is known that the influence of the non-uniformity of wall temperature on the heat transfer rate is more pronounced in laminar than in turbulent flow. In the present investigation, we restrict ourselves to the consideration of laminar, incompressible, two-dimensional boundary layers over wedges of an arbitrary opening angle. The main objective is to develop a procedure that would lead to results by which the aforesaid influence can be readily and accurately ascertained. Because of the linearity of the energy equation, the heart of the problem is to find the solution for a wedge with a step discontinuity in surface temperature.

Tribus and Klein [3] reviewed in 1952 the general problem of heat convection from non-isothermal surfaces and presented a summary of analytical results available at that time. They described an ingenious procedure of finding the wall temperature distribution when the heat flux was prescribed. It made use of an integration kernel which was appropriately modified from that associated with a step discontinuity in surface temperature. Of the more than a dozen investigations reviewed in [3], the one by Levy [4] dealt with incompressible, laminar wedge flows. Consideration was there given to the case in which the wall temperature had a power law variation and the dissipative effects were negligible. Under these conditions, the temperature field is similar and, thus, the analysis becomes greatly simplified. Tribus and Klein mentioned a paper by Bond [5] which is also concerned with wedge flows. To facilitate the solution of the energy boundary layer equation, Bond used a linear velocity distribution. The same approximation was adopted by Lighthill [6] in his analysis of the general problem of heat transfer across a laminar boundary layer with arbitrary distribution of main stream velocity and of wall temperature. Both Lighthill's and Bond's solutions are asymptotically correct for large Prandtl number fluids. For a fixed Prandtl number, the approximation is best for flows without longitudinal pressure gradient. No attempt was made in [3] to discuss the implications of the foregoing approximation.

A number of papers concerning the prediction of heat transfer from non-isothermal surfaces appeared since 1952. They are reviewed in a recent thesis by the junior author [7] and, thus, will not be repeated. Suffice it to state that attempts aiming at improving Lighthill's result have been made by incorporating a more accurate velocity distribution. These attempts have not generally been successful.

2. PROBLEM STATEMENT, GOVERNING EQUATION AND COORDINATE TRANSFORMATION

Consideration is hereby given to the steady, two-dimensional, laminar, incompressible flow over a wedge at sufficiently high Reynolds numbers that the usual boundary layer simplification is valid. The physical model and the coordinate system are shown in Fig. 1. An

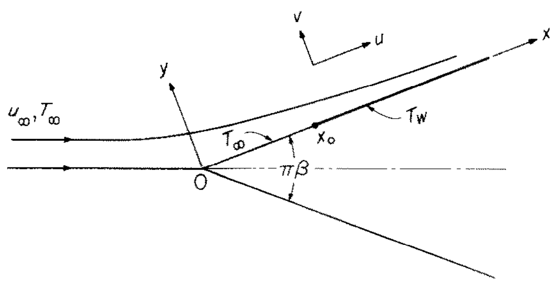

FIG. 1. Physical model and coordinate system.

initial portion of the wedge surface of length x_0 is at the temperature T_∞ of the incoming fluid. The remaining portion of the wedge surface, $x > x_0$, has a uniform temperature T_w which is different from T_∞. The resulting temperature variation is limited so that possible changes in fluid properties are small and may be ignored. Under the foregoing conditions, the thermal problem becomes linear and solutions for any arbitrary surface temperature can be obtained by superposition. Since the dissipative effects, if significant, can be separately assessed, they will not be included in the analysis which follows.

For a wedge of included angle $\pi\beta$, placed symmetrically in a uniform main stream, the velocity u_1 at the edge of the boundary layer is Cx^m, C being a constant and $m = \beta/(2-\beta)$. Falkner and Skan were the first to recognize that the velocity profile in such boundary layer flow is similar and, later, Hartree obtained detailed solutions for the flow field. It is now well known (see e.g. [8]) that the velocity components (u, v) are given by

$$\frac{u}{u_1} = f', \frac{v}{u_1} Re^{\frac{1}{2}} = -(2-\beta)^{-\frac{1}{2}}[f - (1-\beta)\eta f'] \tag{1a,b}$$

where η is the similarity variable defined as

$$\eta = y\left(\frac{1}{2-\beta}\frac{u_1}{\nu x}\right)^{\frac{1}{2}} \tag{2}$$

and the dimensionless stream function $f(\eta)$ satisfies

$$f''' + f'' + \beta[1-(f')^2] = 0 \tag{3}$$

with

$$f(0) = f'(0) = 0; \quad f'(\infty) = 1. \tag{3a}$$

In the foregoing, the primes denote differentiation with respect to η. The power series solution for f is

$$f = \sum_{n=2}^{\infty} \frac{a_n}{n!}\eta^n \tag{4}$$

with

$$a_2 = a, a_3 = -\beta, a_4 = 0, a_5 = (2\beta-1)a^2,$$
$$a_6 = -2(3\beta-2)\beta a, a_7 = 2(3\beta-2)\beta^2, \text{etc.} \tag{4a}$$

Numerical values of a for various β have been extensively tabulated; the more recent ones are by Elzy and Sisson [8].

Under the assumptions previously stated, the energy boundary layer equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} \tag{5}$$

and its solution which we seek must satisfy the following entrance and boundary conditions

$$T(x_0, y > 0) = T_\infty \tag{5a}$$

$$T(x > x_0, 0) = T_w, T(x,\infty) = T_\infty. \tag{5b,c}$$

Because of the presence of the reference length x_0, the temperature field is, in general, non-similar.

To facilitate analysis, we introduce a dimensionless temperature function,

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{6}$$

and a coordinate transformation,

$$\left.\begin{matrix} x \\ y \end{matrix}\right\} \Rightarrow \begin{cases} X = \left[1 - \left(\dfrac{x_0}{x}\right)^c\right]^{\frac{1}{3}} & (7a) \\ \xi = b\dfrac{\eta}{X}. & (7b) \end{cases}$$

Clearly, both X and ξ are dimensionless. The specific forms chosen are motivated from the following considerations.

(i) In anticipation of developing a series solution in powers of X, it would be desirable to have the range of X restricted. Equation 7(a) requires that X is bounded between 0 and 1.

(ii) The approximation of using a linear velocity distribution corresponds to retaining only the first term of the series for f; i.e. $f = (1/2)a\eta^2$. Under this condition, the temperature field becomes self-similar [5]. For later reference, it will be designated as the *reduced problem*. The transformed coordinate ξ defined in (7b) is the similarity variable of the reduced problem.

(iii) In order that the end result of the analysis has general applicability and is simple to use, individual terms in the series solution should only comprise of functions that are universal, i.e. they can be tabulated once and for all. Furthermore, it is desirable, though not necessary, that the dominant terms of the series can be expressed in closed form, thus rendering possible an analytical description of the essential features of the solution.

Using (6) and (7a,b), (5) becomes

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{Pr}{b}\left[Xf(\eta) + \frac{1 - X^3}{2bX}\xi f'(\eta)\right]\frac{\partial\theta}{\partial\xi} - \frac{Pr}{2b^2}(1 - X^3)f'(\eta)\frac{\partial\theta}{\partial X} = 0 \tag{8}$$

with

$$\theta(X,0) = 1, \quad \theta(X,\infty) = 0. \tag{9a,b}$$

In (8), $0 \leqslant X \leqslant 1$ and $0 \leqslant \xi < \infty$; the argument η of the stream function is related to ξ according to (7b). Because of the form chosen for ξ, the entrance condition (5a) merges into (9b).

If the entire wedge surface has a uniform temperature, then $x_0 = 0$ and, hence, $X = 1$ and $\xi = b\eta$. When this occurs, θ depends only on η and (8) and (9a,b) reduce to the following familiar forms.

$$\frac{d^2\theta_{iso}}{d\eta^2} + Prf\frac{d\theta_{iso}}{d\eta} = 0 \tag{10}$$

with

$$\theta_{iso}(0) = 1, \quad \theta_{iso}(\infty) = 0. \tag{11a,b}$$

The subscript iso refers to the isothermal surface condition. The solution of (10) satisfying (11a,b) has been extensively studied. A recent tabulation of the wall derivatives $\theta'_{iso}(0)$ for wide ranges of Pr and β may be found in [9].

3. SOLUTION METHOD AND RESULTS

We seek a series solution for (8) of the form

$$\theta = \sum_{n=0}^{\infty} F_n(\xi)X^n \tag{12}$$

with

$$F_0(0) = 1; \quad F_1(0) = F_2(0) = \ldots = 0 \tag{13a}$$

and

$$F_0(\infty) = F_1(\infty) = \ldots = 0. \tag{13b}$$

Hence, the boundary conditions (9a, b) are satisfied. Upon substituting (12) into (8) and comparing the coefficients of like powers of X, we obtain a sequence of second order, linear, ordinary differential equations which are given below (the primes denote differentiation with respect to ξ).

$$F_0'' + 3\xi^2 F_0' = 0 \tag{14a}$$

$$F_1'' + 3\xi^2 F_1' - 3\xi F_1 = g_1 F_0' \tag{14b}$$

$$F_2'' + 3\xi^2 F_2' - 6\xi F_2 = g_1 F_1' - h_1 F_1 \tag{14c}$$

etc. In general, for $n \geqslant 1$

$$F_n'' + 3\xi^2 F_n' - 3n\xi F_n = g_n F_0' + g_{n-1}F_1' + \ldots + g_1 F_{n-1}' - h_{n-1}F_1 - 2h_{n-2}F_2 - \ldots - (n-1)h_1 F_{n-1} \tag{14}$$

wherein

$$\left.\begin{aligned} g_1 &= -\frac{3}{2}\frac{a_3}{ab}\xi^3, \quad g_2 = 0, \\ g_3 &= -\frac{3}{4}\frac{a_5}{a^2 Pr}\xi^5, \\ g_4 &= -\frac{18}{5!}\frac{a_6}{a^2 bPr}\xi^6 + \frac{a_3}{2ab}\xi^3, \\ g_5 &= -\frac{3}{5!}\frac{a_7}{a^2 b^2 Pr}\xi^7, \\ &\text{etc. and} \\ h_1 &= \frac{g_1}{\xi}, \quad h_2 = 0, \quad h_3 = 3\xi + \frac{g_3}{\xi} \\ h_4 &= \frac{1}{\xi}\left(g_4 - \frac{g_1}{3}\right), \text{ etc.} \end{aligned}\right\} \tag{15}$$

Equation (14a) with the assigned boundary conditions, $F_0(0) = 1$ and $F_0(\infty) = 0$, can be integrated in closed form. The solution is

$$F_0 = 1 - \frac{\Gamma[(1/3), \xi^3]}{\Gamma(1/3)} \tag{16}$$

and

$$F_0'(0) = -\frac{3}{\Gamma(1/3)} = -1{\cdot}1198. \tag{17}$$

An examination of the equation set (14) in conjunction with (15) reveals that, if all a_n's vanish, except a_2, then all F_n's other than F_0 would vanish. This is the case when $f = (1/2)a\eta^2$. Hence, the solution of the reduced problem previously defined is simply

$$\theta = F_0 \tag{18}$$

and the corresponding wall heat flux is

$$q_w = -k(T_w - T_\infty)\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial\eta}F_0'(0) = 0{\cdot}6163k(T_w - T_\infty)\left(\frac{1}{2-\beta}\frac{u_1}{\nu x}\right)^{1/2}(aPr)^{1/3}X^{-1}. \tag{19}$$

These are precisely Bond's results [5], although they were obtained by Bond from a totally different procedure. Since the local friction coefficient is

$$c_f = 2\nu\frac{1}{u_1^2}\frac{\partial u}{\partial y}(0) = 2\left(\frac{1}{2-\beta}\frac{\nu}{u_1 x}\right)^{\frac{1}{2}} a, \tag{20}$$

(19) may be expressed in terms of the local Stanton number as

$$St\,Pr^{\frac{2}{3}}\frac{2}{c_f} = 0{\cdot}6163a^{-\frac{2}{3}}X^{-1}. \tag{21}$$

For a flat plate, $\beta = 0$, $a = 0{\cdot}4696$, $X = [1 - (x_0/x)^{\frac{3}{4}}]^{\frac{1}{3}}$ and, hence, (21) reduces to

$$St\,Pr^{\frac{2}{3}}\frac{2}{c_f} = \frac{1{\cdot}020}{[1 - (x_0/x)^{\frac{3}{4}}]^{\frac{1}{3}}} \tag{22}$$

which becomes identical to Eckert's result [10] if the numerical constant 1·020 is replaced by unity. Eckert deduced his expression from the integral heat balance equation, using third-degree polynomials for both the velocity and temperature profiles. Considering the fact that (22) is obtained for linear velocity distribution, the close agreement is probably fortuitous.

For large Prandtl number fluids, the thermal boundary layer is everywhere thin relative to the velocity boundary layer. Under such circumstance, (18), (19) and (21) may be expected to yield very satisfactory results. We shall return to this point later.

The equation for F_1 can also be integrated in closed form. Since $g_1 = (3/2)(\beta/ab)\xi^3$ and $F_0' = -[3/\Gamma(1/3)]e^{-\xi^3}$, (14b) can be written as

$$F_1'' + 3\xi^2F_1' - 3\xi F_1 = -\frac{9}{2\Gamma(1/3)}\frac{\beta}{ab}\xi^3 e^{-\xi^3}. \tag{23}$$

If a related function $\bar{F}_1$ is defined such that

$$F_1 = M\bar{F}_1 \tag{24}$$

with

$$M = \frac{3}{2}\frac{\beta}{ab} \tag{25}$$

then

$$\bar{F}_1'' + 3\xi^2\bar{F}_1' - 3\xi\bar{F}_1 = -\frac{3}{\Gamma(1/3)}\xi^3 e^{-\xi^3} \tag{26}$$

with $\bar{F}_1(0) = \bar{F}_1(\infty) = 0$. Clearly, $\bar{F}_1$, like F_0, depends only on ξ and, thus, can be tabulated once and for all. For this reason, they will be referred to as universal functions.

An equivalent form of (26) is

$$\left(\frac{\bar{F}_1}{\xi}\right)'' + \left(\frac{2}{\xi} + 3\xi^2\right)\left(\frac{\bar{F}_1}{\xi}\right)' = -\frac{3}{\Gamma(1/3)}\xi^2 e^{-\xi^3} \tag{27}$$

which, upon integrating twice and using the stated boundary conditions, yields

$$\bar{F}_1 = \frac{1}{5\Gamma(1/3)}\xi[\Gamma(4/3) - \Gamma(4/3, \xi^3)]. \tag{28}$$

It follows that

$$\bar{F}_1'(0) = \frac{1}{15}. \tag{29}$$

An inspection of the equation for F_2 discloses that it, too, can be rewritten in terms of a universal function $\bar{F}_2$ defined by

$$F_2 = M^2\bar{F}_2. \tag{30}$$

The fourth function F_3 can be expressed as a linear combination of two universal functions $\bar{F}_{3,1}$ and $\bar{F}_{3,2}$ according to

$$F_3 = M^3\bar{F}_{3,1} + \frac{1-2\beta}{Pr}\bar{F}_{3,2}. \tag{31}$$

Table 1. Universal functions

ξ	F_0	$\bar{F}_1 \times 10$	$\bar{F}_2 \times 10^2$	$\bar{F}_{3,1} \times 10^2$	$\bar{F}_{3,2} \times 10$	$\bar{F}_{6,1}$† $\times 10^2$	$\bar{F}_{6,2}$† $\times 10^2$
0·0	1·00000	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·88804	0·06666	0·08177	0·01721	0·02075	0·04140	−0·01046
0·2	0·77648	0·13316	0·16382	0·03453	0·04164	0·08136	−0·02110
0·3	0·66630	0·19867	0·24688	0·05221	0·06301	0·11628	−0·03240
0·4	0·55910	0·26116	0·33209	0·07060	0·08528	0·14078	−0·04523
0·5	0·45697	0·31706	0·42073	0·09012	0·10866	0·14881	−0·06082
0·6	0·36225	0·36147	0·51350	0·11120	0·13274	0·13567	−0·08061
0·7	0·27725	0·38899	0·60917	0·13425	0·15603	0·10015	−0·10542
0·8	0·20386	0·39506	0·70281	0·15949	0·17580	0·04626	−0·13406
0·9	0·14328	0·37765	0·78437	0·18668	0·18850	−0·01671	−0·16152
1·0	0·09574	0·33843	0·83922	0·21455	0·19073	−0·07615	−0·17859
1·1	0·06049	0·28298	0·85157	0·24016	0·18060	−0·11996	−0·17433
1·2	0·03593	0·21967	0·81056	0·25871	0·15877	−0·14056	−0·14166
1·3	0·01995	0·15747	0·71627	0·26439	0·12868	−0·13732	−0·08325
1·4	0·01029	0·10369	0·58216	0·25259	0·09552	−0·11606	−0·01312
1·5	0·00491	0·06238	0·43160	0·22249	0·06453	−0·08609	0·04866
1·6	0·00216	0·03409	0·28971	0·17851	0·03945	−0·05630	0·08583
1·7	0·00088	0·01683	0·17486	0·12909	0·02170	−0·03240	0·09356
1·8	0·00033	0·00746	0·09430	0·08339	0·01068	−0·01625	0·07890
1·9	0·00013	0·00296	0·04515	0·04774	0·00467	−0·00688	0·05465
2·0		0·00104	0·01907	0·02405	0·00181	−0·00218	0·03180
2·1		0·00032	0·00706	0·01058	0·00062	−0·00011	−0·01573
2·2			0·00227	0·00404	0·00018		0·00667
2·3			0·00063	0·00133			0·00249
2·4			0·00015	0·00037			0·00089

† $\bar{F}_{6,1}$ and $\bar{F}_{6,2}$ valid for $\beta = 0$ only.

The associated boundary conditions are, respectively, $\bar{F}_2(0) = \bar{F}_2(\infty) = 0$; $\bar{F}_{3,1}(0) = \bar{F}_{3,1}(\infty) = 0$ and $\bar{F}_{3,2}(0) = \bar{F}_{3,2}(\infty) = 0$. The resulting differential equations, however, have not been solved in closed forms. They were integrated numerically, using routine iterative techniques on the computer with a uniform step size of $\Delta\xi = 0{\cdot}05$. Some details may be found in [7]. The results, along with other universal functions, are given in Table 1*. They are also shown graphically in Figs. 2a and 2b. The associated wall derivatives are

$$\bar{F}'_2(0) = 0{\cdot}81748 \times 10^{-2} \tag{32}$$

$$\bar{F}'_{3,1}(0) = 0{\cdot}17204 \times 10^{-2} \quad \text{and}$$
$$\bar{F}'_{3,2}(0) = 0{\cdot}20737 \times 10^{-1}. \tag{33a, b}$$

* More complete data with ξ at multiples of 0·05, beginning at 0 and extending to 3·95, are on deposit in the Heat Transfer Laboratory, Department of Mechanical and Industrial Engineering.

To check the accuracy of the numerical procedure, the same computer program was used to evaluate F_0, $F'_0(0)$, $\bar{F}_1$ and $\bar{F}'_1(0)$ and the data were compared with the respective closed form solutions given earlier. Values of the gamma functions which appear in (16) and (28) were taken from [11]. It was found that the wall derivatives were in agreement up to the fifth significant figure and all values of F_0 and $\bar{F}_1$ agree to at least four significant figures for the entire range of ξ.

A study of the general equation (14) shows that it is always possible to express the F_n functions as linear combinations of universal functions. The resulting second order, ordinary differential equations can be routinely handled on the computer. However, for fluids of Pr of the order of unity and larger, highly accurate results are obtainable with the several functions already evaluated. An estimation of the maximum error resulting from retaining a finite

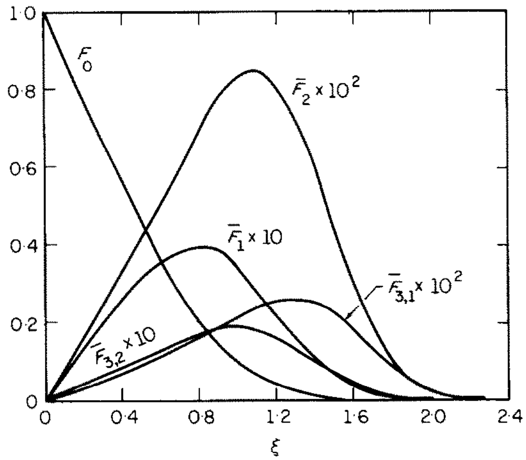

FIG. 2a. Universal functions for wedges with arbitrary opening angle.

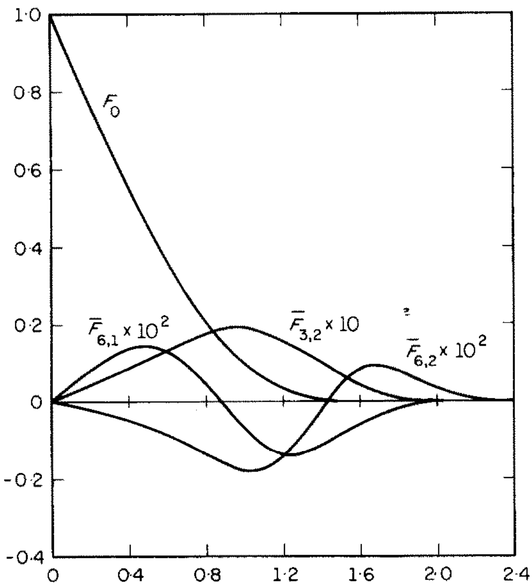

FIG. 2b. Universal functions for a semi-infinite flat plate.

number of terms in the series (12) will be presented in Section 4.

With the information provided in Table 1 and the values of the wall derivative listed, the determination of the temperature field in the boundary layer and the local wall heat flux for wedges of any arbitrary opening angle becomes a matter of simple arithmetic. Before proceeding further, we summarise the results as follows

(a) *Temperature field in boundary layer*

$$\theta(X, \xi) = \sum_{n=0}^{\infty} F_n(\xi) X^n$$

$$= F_0 + M\bar{F}_1 X + M^2 \bar{F}_2 X^2$$

$$+ \left(M^3 \bar{F}_{3,1} + \frac{1 - 2\beta}{Pr} \bar{F}_{3,2} \right) X^3 + \dots \qquad (34)$$

(b) *Heat flux at wall*

$$\frac{q_w}{c_p \rho u_1 (T_w - T_\infty)} = \frac{c_f}{2} Pr^{-\frac{2}{3}} (3! a^2)^{-\frac{1}{3}} X^{-1} \left[-\frac{\partial \theta}{\partial \xi}(X, 0) \right] \qquad (35)$$

where

$$\frac{c_f}{2} = \left(\frac{1}{2 - \beta} \frac{\nu}{u_1 x} \right)^{\frac{1}{2}} a \qquad (35a)$$

and

$$-\frac{\partial \theta}{\partial \xi}(X, 0) = -\sum_{n=0}^{\infty} F_n'(0) X^n$$

$$= 1{\cdot}1198 - \frac{1}{15} MX - 0{\cdot}81748 \times 10^{-2} M^2 X^2$$

$$- \left(0{\cdot}17204 \times 10^{-2} M^3 + 0{\cdot}20737 \times 10^{-1} \frac{1 - 2\beta}{Pr} \right) X^3 + \dots \qquad (35b)$$

In the above, $M = (3/2)(\beta/ab) = 2.7257 \beta Pr^{-\frac{1}{3}} a^{-\frac{4}{3}}$ and $X = [1 - (x_0/x)^{1{\cdot}5/(2-\beta)}]^{\frac{1}{3}}$. The ξ-coordinate is related to y through (2) and (7b).

The case of longitudinal flow past a semi-infinite flat plate ($\beta = 0$) is of particular interest, both from the theoretical and practical point of view. In this instance, F_1 and F_2 are identically zero; so are F_4 and F_5. Moreover, since $M = 0$, $F_3 = Pr^{-1} \bar{F}_{3,2}$. To provide one more non-vanishing term in the series solution, we have evaluated F_6. The latter is expressible as a combination of two universal functions as according to

$$F_6 = Pr^{-1}\bar{F}_{6,1} + Pr^{-2}\bar{F}_{6,2}. \quad (36)$$

Data for these two universal functions are also included in Table 1. Their wall derivatives are

$$\bar{F}'_{6,1}(0) = 0{\cdot}41502 \times 10^{-2} \quad \text{and}$$
$$\bar{F}'_{6,2}(0) = -0{\cdot}10445 \times 10^{-2}. \quad (37a, b)$$

Hence, for this case of flow with zero pressure gradient, the temperature field is

$$\theta(X, \xi) = F_0 + Pr^{-1}\bar{F}_{3,2}X^3 + Pr^{-1}(\bar{F}_{6,1} + Pr^{-1}\bar{F}_{6,2})X^6 + \ldots \quad (38)$$

and the wall flux is

$$\frac{q_w}{c_p\rho u_\infty(T_w - T_\infty)} = 0{\cdot}30247 Re^{-\frac{1}{2}}\, Pr^{-\frac{2}{3}}\, X^{-1} \left[-\frac{\partial\theta}{\partial\xi}(X, 0)\right] \quad (39)$$

wherein

$$-\frac{\partial\theta}{\partial\xi}(X, 0) = 1{\cdot}1198 - 0{\cdot}20737 \times 10^{-1} Pr^{-1} X^3 - Pr^{-1}(0{\cdot}41502 - 0{\cdot}10445 Pr^{-1}) \times 10^{-2} X^6 + \ldots \quad (39a)$$

and

$$X = [1 - (x_0/x)^{\frac{3}{4}}]^{\frac{1}{3}},\ \xi = 0{\cdot}30247 Pr^{\frac{1}{3}}(y/x)Re^{\frac{1}{2}}\, X^{-1}$$

The development of the temperature profile in the boundary layer, downstream of the surface temperature discontinuity, can be readily obtained from (34) or (38). They are illustrated in Figs. 3a, b and c, respectively, for a flat plate ($\beta = 0$), a wedge with 90 degree included angle ($\beta = 0{\cdot}5$) and the stagnation flow condition ($\beta = 1{\cdot}0$). In each figure, groups of profiles are shown for $Pr = 1$, 10 and 100 and for three locations along the wedge surface. The results show that at $x_0/x = 0{\cdot}95$, i.e. a location at which the unheated length is 95 per cent of its distance from the leading edge, considerable development of the thermal boundary layer has already taken place. At $x_0/x = 0{\cdot}05$ the development is, for all practical purposes, complete. In fact, if the fully developed similar profiles were plotted, they would fall on those for $x_0/x = 0{\cdot}05$ within the width of the line.

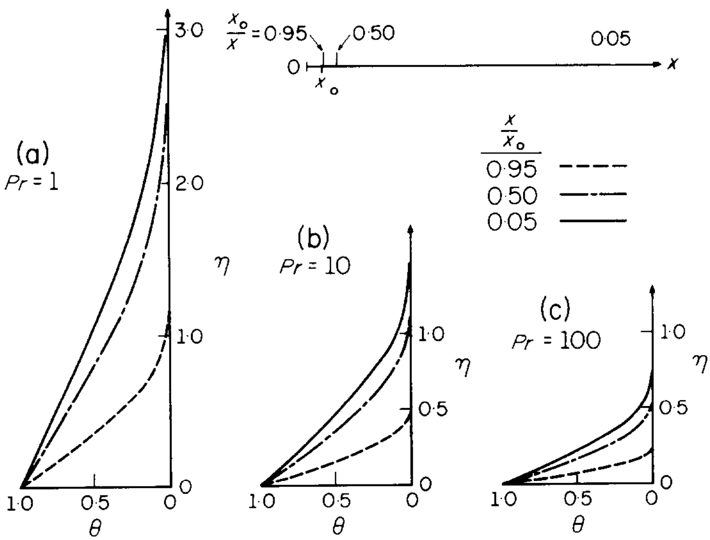

FIG. 3a. Development of temperature profile downstream of the step discontinuity in surface temperature—$\beta = 0$.

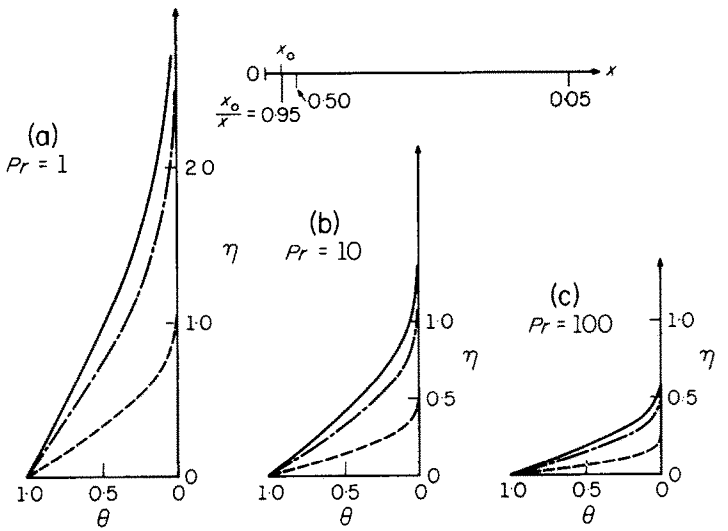

FIG. 3b. Development of temperature profile downstream of the step discontinuity in surface temperature—$\beta = 0{\cdot}5$.

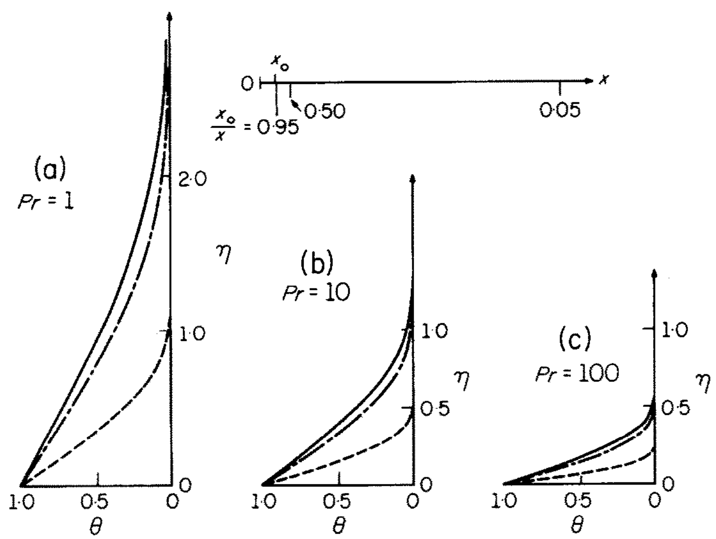

FIG. 3c. Development of temperature profile downstream of the step discontinuity in surface temperature—$\beta = 1{\cdot}0$.

4. ACCURACY OF RESULTS

As has been pointed out earlier, the first term of the series in (34) represents the mathematically exact solution to the problem when the velocity distribution is linear. The rest may be regarded as corrections for the departure of the actual velocity profile from the linear distribution. Within a short distance downstream of the surface temperature discontinuity, the thermal boundary layer is thin and the corrections are small, regardless of the Prandtl number. On the other hand, at sufficiently large distance downstream where the thermal boundary layer has had significant development, its thickness relative to the velocity boundary layer depends on the Prandtl number. The effect on the temperature field due to the nonlinearity in the velocity distribution will manifest itself to the fullest when $X = 1$. Thus, the maximum error resulting from using a finite number of terms in the series occurs at $X = 1$. A portion of such error is intimately connected with the inaccuracy inherent in the representation of the Falkner–Skan stream function by the power series (4) which has a limited radius of convergence.

It is clear that, when $x_0 = 0$, $X = 1$ for all x's > 0. This is the situation when the entire wedge surface has a uniform temperature. Hence, by setting $X = 1$ in (34) and comparing the results to those obtained from integrating (10) with boundary conditions (11a, b), an *upper bound* of the errors for the entire range of ξ can be established. To this end, both (3) and (10) were numerically integrated, using a scheme essentially identical to that for evaluating $\bar{F}_2$, $\bar{F}_{3,1}$, $\bar{F}_{3,2}$, etc. Since the wall derivatives $f''(0)$ and $\theta'_{iso}(0)$ are known, the integration involves no iteration. In our computation, the wall derivative data were taken from [9].

If one denotes the said upper bound of error by $(\delta\theta)_u$, then

$$(\delta\theta)_u = \theta(1, \xi)_{t.s.} - \theta_{iso} \tag{40}$$

where $\xi = b\eta$. The subscript t.s. designates truncated series. Clearly, $(\delta\theta)_u$ depends on ξ; it vanishes at $\xi = 0$ and as $\xi \to \infty$. The particular value of ξ at which $(\delta\theta)_u$ exhibits an extremal depends on Pr and β. A summary of the results of an extensive calculation is compiled in Table 2 which lists the extremes of the upper bound error for the entire range of Pr and β investigated. Included for the purpose of comparison are error data evaluated from Bond's equation which is for the linear velocity distribution and corresponds to the first term of our series. As

Table 2. Extremes of upper bound error in temperature field, $(\delta\theta)_{u,\,extremal}$

β \ Pr	0.72		1		10		100	
	Present	Bond	Present	Bond	Present	Bond	Present	Bond
−0·1	0·01260	0·02158	0·00594	0·02102	−0·00152	0·01746	−0·00023	0·00297
0·0	0·00339	−0·01950	0·00248	−0·01470	0·00024	−0·00165	0·00002	0·00017
0·2	0·01330	−0·05186	0·00862	−0·04577	0·00205	−0·01705	0·00077	0·00732
0·4	0·01537	−0·07255	0·01185	−0·06330	0·00314	−0·02530	0·00119	0·01099
0·6	0·02028	−0·08492	0·01481	−0·07554	0·00382	−0·03117	0·00155	0·01346
0·8	0·02312	−0·09602	0·01747	−0·08521	0·00494	−0·03528	0·00176	0·01545
1·0	0·02609	−0·10540	0·02036	−0·09305	0·00519	−0·03931	0·00199	0·01705

Table 3. Upper bound error in wall temperature derivative $1 - \theta'(1,0)_{t.s.}/\theta'_{iso}(0)$

β \ Pr	0·72		1		10		100	
	Present	Bond	Present	Bond	Present	Bond	Present	Bond
−1·0	−0·00361	0·03691	−0·00654	0·03567	−0·00197	0·02734	−0·00096	0·01412
0·0	0·00300	−0·02683	0·00166	−0·02005	0·00001	−0·00221	0·00000	−0·00022
0·2	0·00817	−0·08292	0·00043	−0·07261	0·00129	−0·02680	0·00059	−0·01149
0·4	0·01047	−0·11948	0·00777	−0·10333	0·00221	−0·04017	0·00092	−0·01731
0·6	0·01518	−0·14270	0·01046	−0·12548	0·00273	−0·04977	0·00124	−0·02127
0·8	0·01674	−0·16364	0·01237	−0·14337	0·00377	−0·05668	0·00139	−0·02444
1·0	0·01771	−0·18154	0·01412	−0·15822	0·00383	−0·06332	0·00156	−0·02701

has been explained earlier, the use of a linear velocity profile would lead to satisfactory results when Pr is large. This is indeed borne out by the data.

By following a similar line of reasoning, one is also led to conclude that the upper bound error in the wall heat flux resulting from using a finite number of terms in (35b) or (39a) would occur when $X = 1$. Since the numerical values of the wall temperature derivative vary with Pr, it is convenient to express such error in a ratio defined as

$$1 - \frac{\theta'(1,0)_{t.s.}}{\theta'_{iso}(0)}. \tag{41}$$

Table 3 provides this information. It is seen that, for $Pr = 1$, Bond's expression entails an upper bound error of 15·8 per cent when $\beta = 1$, while the corresponding error in using four terms of our series is 1·4 per cent. The exceptionally high accuracy of the flat plate data ($\beta = 0$), evaluated from the present, as well as from Bond's expression, as shown in both Tables 2 and 3, is mainly due to the fact that, in the absence of longitudinal pressure gradient, the curvature of the velocity profile vanishes at the wall. In addition, the series expressing $(\partial\theta/\partial\xi)(1,0)$ as given in (39a) for the flat plate has effectively more terms than the one given in (35b).

Recently, Clausing [12] reported wall flux data for longitudinal flow past a flat plate and for $Pr = 0{\cdot}72$ and 1·0. The data were obtained by directly solving the governing conservation equations for the boundary layer flow, using an all numeric, finite difference technique. A comparison of the data evaluated from (39) and (39a) with those of Clausing for x_0/x ranging from 0·05 to 0·909 reported by him shows that in no case the discrepancy exceeds 0·4 per cent. Clausing estimated that the absolute error in his data was less than 0·5 per cent for $0{\cdot}1 < (x_0/x) < 0{\cdot}9$. Hence, all evidence demonstrates beyond doubt that the main results of the present analysis—(34) or (38) for the temperature field; (35) or (39) for the wall flux—are highly accurate for fluids of Prandtl number comparable to or greater than that of air. They are also simple to use. However, similar accuracy is not likely to obtain for smaller Prandtl number fluids. An inherent error in the present analysis is the representation of the velocity function f by the power series (4) which has a limited radius of convergence. This source of error is negligible in high or moderately high Prandtl number fluids since the thermal boundary layer is everywhere confined to a relatively thin region adjacent to the solid surface. This is not the case when the Prandtl number is small.

An inspection of (34), (35b), (38) and (39a) shows that, when Pr is of the order of 10^{-2}–10^{-3}, the series does not converge except when X is small. When this occurs, one may tacitly assume that they are semi-divergent and Euler's transformation may be used for the evaluation of the sum. Based on our own experience and that reported in [13], the following procedure is tentatively suggested.

To begin with, more terms in the series should be computed. Since the exact solution for the problem is available for a wedge with uniform surface temperature and without the restriction on Pr, the optimum number of terms to be retained in the series and the best starting point for the application of the Euler transformation can be ascertained by comparing the results obtained for $X = 1$ with the corresponding exact solution. The same number of terms is then used in evaluating the sum of the series when $X \neq 1$ with the Euler transformation applied in precisely the same manner.

5. CONCLUDING REMARKS

With the information provided in Table 1 and the values of the wall derivatives of the universal functions given in the text, problems with any arbitrarilly prescribed surface temperature variation can be handled in a straightforward manner either formally via the Duhamel's integral or by numerical superposition. We refrain from giving the details since they are well documented in the literature. It is our hope that the modus operandi of the analysis described in this paper could be extended to treat general two-dimensional and axisymmetrical boundary layer flows. If this can be achieved, then the Lighthill analysis is truly improved. It is also conceivable that the highly accurate results presented in the paper could usefully serve as a comparison standard in the study of trial solutions associated with integral methods as described by Walz [14].

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CONVECTION FORCÉE D'UN ÉCOULEMENT AUTOUR D'UN DIÈDRE POUR DES SURFACES NON ISOTHERMES

Résumé—On étudie le transfert thermique à travers des couches limites laminaires incompressibles à propriétés constantes sur des dièdres avec une température pariétale en échelon. L'analyse débute par une transformation appropriée de l'équation d'énergie de la couche limite avec pour conséquence que la solution du problème se décompose en une suite infinie de solutions simples réduites. L'intérêt en est que ces solutions réduites sont exprimables comme des fonctions universelles et par suite peuvent être tabulées une fois pour toutes. Avec des fluides ayant des nombres de Prandtl de l'ordre de l'unité ou plus, seul un petit nombre de fonctions sont nécessaires pour obtenir des résultats avec précision. Une tabulation de ces fonctions est donnée. La détermination du champ de température dans la couche limite aussi bien que le flux pariétal local est une affaire d'arithmétique élémentaire. Une information semblable pour des dièdres avec une distribution de température superficielle donnée quelconque peut être obtenue de la même façon.

ERZWUNGENE KONVEKTION BEI KEILSTRÖMUNG MIT NICHTISOTHERMEN OBERFLÄCHEN

Zusammenfassung— Es wird die stationäre Wärmeübertragung in laminaren, inkompressiblen Grenzschichten mit konstanten Stoffeigenschaften über Keile mit stufenförmigem Sprung in der Oberflächentemperatur untersucht. Die Analysis beginnt mit einer geeigneten Transformation der Energiegleichung der Grenzschicht, aus der folgt, dass die nichtähnliche Lösung des Problems in eine unendliche Folge von einfachen ähnlichen Lösungen zerfällt. Wichtig ist die Tatsache, dass sich diese ähnlichen Lösungen als universelle Funktionen ausdrücken lassen und deshalb ein für allemal tabelliert werden können. Für Flüssigkeiten mit Prandtl–Zahlen in der Grössenordnung von eins oder grösser werden nur einige wenige dieser Funktionen gebraucht, um Ergebnisse mit hoher Genauigkeit zu erzielen. Einige Tabellen für solche Funktionen werden angegeben. Mit diesen wird die Bestimmung des Temperaturfeldes in der Grenzschicht ebenso wie die Bestimmung der örtlichen Wärmestromdichte an der Keiloberfläche mit einfachen arithmetischen Mitteln möglich. Ähnliche Aussagen kann man in gleicher Weise für Keile mit irgendeiner willkürlich vorgeschriebenen Oberflächentemperaturverteilung erhalten.

ВЫНУЖДЕННАЯ КОНВЕКЦИЯ ДЛЯ КЛИНОВЫХ ТЕЧЕНИИ С НЕИЗОТЕРМИЧЕСКИМИ ПОВЕРХНОСТЯМИ

Аннотация—Исследовался процесс установившегося теплообмена в ламинарных, несжимаемых пограничных слоях с постоянными свойствами при обтекании поверхности со ступенчатым распределением температуры стенки. Анализ начинается с соответствующего преобразования уравнения теплового пограничного слоя, такого, что сформулированная неавтомодельная задача может быть представлена бесконечной совокупностью простых, автомодельных решений. Здесь важно то обстоятельство, что эти составляющие автомодельные решения выражаются через универсальные функции и, таким образом, могут быть табулированы раз и навсегда. Для получения результатов высокой точности при числах Прандтля порядка единицы или выше требуется небольшое число универсальных функций. Эти функции приведены в таблице. С их помощью определение температурного поля в пограничном слое, так же как и величины локального коэффициента теплообмена на поверхности клина, сводится к простой арифметической операции. Сходные расчёты можно провести для клинов с любым произвольным распределением заданной температуры поверхности.